3^{rd} ASSIGNMENT

DUE DATE: FRIDAY, JUNE 24^{TH}

1. Differential Equations and Exponential Growth/Decay

Problem 1. The isotope thorium-234 has a half-life of 24.5 days.

(a) What is the differential equation satisfied by $y(t)$, the amount of thorium-234 in a sample at time t?

(b) At $t = 0$, a sample contains 2 kg of thorium-234. How much remains after 40 days?

Problem 2. Verify that if the function $f(x)$ is a solution to the corresponding differential equation:

(a) $f(x) = 4e^{3x}\sin x,\ y' = 3y + 4e^{3x}\cos x$

(b) $f(x) = e^{3x},\ y'' + 2y' \cdot 15y = 0$

(c) $f(x) = e^{x},\ y'' = x$

(d) $f(x) = \frac{2x^2}{2x-3},\quad y' = \frac{y^2(x-3)}{x^3}$

2. Integration by Substitution

Problem 3. Compute the following indefinite integrals, using the suggested substitution.

(a) $\int \sec^2 x \tan x\, dx,\ u = \tan x$

(b) $\int xe^{-x^2} dx,\ u = -x^2$

(c) $\frac{(\arctan x)^2}{x^2+1} dx,\ u = \arctan x$

(d) $\int \sin^2\theta \cos\theta\, d\theta,\ u = \sin\theta$

(e) $\int \sin(4\theta - 7) d\theta,\ u = 4\theta - 7$

(f) $\int x^2\sqrt{x+1}\, dx,\ u = x + 1$

Problem 4. Compute the following indefinite integrals:

(a) $\int (x^2 + 1)(x^3 + 3x)^4 dx$

(b) $\int \cot x \csc 2x\, dx$

(c) $\int \frac{\cos(\pi/x)}{x^2} dx$

(d) $\int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$

(e) $\int xe^{1-x} dx$

(f) $\int x^3\sqrt{x^2+1}\, dx$

(g) $\int x^2\sqrt{2+x}\, dx$

(h) $\int \frac{(\ln u)^4}{u} du$

(i) $\int \frac{dx}{(1+\sqrt{x})^4}$

(j) $\int x^{-1/5}\tan(x^{4/5})dx$

Problem 5. Solve the following differential equations with given initial value:

(a) $y' = \frac{\cos x}{\sin^2 x}$, $y(\frac{\pi}{2}) = 3$
(b) $y' = \frac{2x}{\sqrt{1-x}}$, $y(0) = 0$
(c) $y' = \frac{dx}{\cos^2 x\sqrt{1+tanx}}$, $y(\frac{\pi}{4}) = 7$
(d) $y' = \frac{\cos x}{\sin^2 x}dx$, $y(-\frac{\pi}{4}) = 0$
(e) $y' = e^x\sqrt{1-e^x}$, $y(-\ln 2) = 0$
(f) $y' = x(2x+5)^8$, $y(1) = -2$

Problem 6. *Can They Both Be Right?* Hannah uses the substitution $u = \tan x$ and Akiva uses $u = \sec x$ to evaluate $\int \tan x \sec^2 x dx$. Show that they obtain different answers, and explain the apparent contradiction.

Problem 7. Evaluate $\int \sin x \cos x\, dx$ using substitution in two different ways: first using $u = \sin x$ and then using $u = \cos x$. Reconcile the two different answers.

Problem 8. *Some Choices Are Better Than Others.* Evaluate

$$\int \sin x \cos^2 x\, dx$$

twice. First use $u = \sin x$ to show that

$$\int \sin x \cos^2 x\, dx = \int u\sqrt{1-u^2}\, du$$

and evaluate the integral on the right by a further substitution. Then show that $u = \cos x$ is a better choice.

3. Fundamental Theorem of Calculus, part II

Problem 10. Find the smallest positive critical point of

$$F(x) = \int_0^x \cos(t^{3/2})dt$$

and determine whether it is a local min or max.

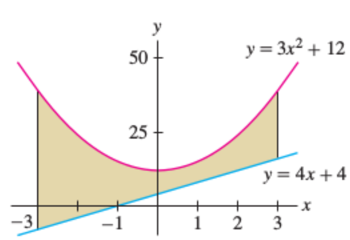
Problem 11. Evaluate

$$\frac{d}{dx}\int_{\ln x}^{e^x} \sin t\, dt\ .$$

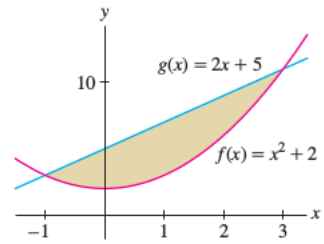
4. Area Between Curves

Problem 12.

(a) Find the area of the region between $y = 3x^2 + 12$ and $y = 4x + 4$ over $[-3, 3]$:



(b) Find the area of the region enclosed by the graphs of $f(x) = x^2 + 2$ and $g(x) = 2x + 5$:



Problem 13. Sketch the region bounded by

$$y = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad y = -\frac{1}{\sqrt{1-x^2}}$$

for $-\frac{1}{2} \leq x \leq \frac{1}{2}$ and find its area.

Problem 14. Sketch the region bounded by the curves and compute its area:

(a) $y = x + 1$, $y = 9 - x^2$, $x = -1$ and $x = 2$
(b) $y = \frac{1}{x^2}$, $y = x$ and $x = \frac{1}{8}x$
(c) $y = 3x^2$, $y = 8x^2$, $4x + y = 4$ and $x \geq 0$

5. The Method of Partial Fractions

Problem 15. Write out the form of the partial fraction decomposition of the function:

(a) $\frac{2x}{(x+3)(3x+1)}$
(b) $\frac{1}{x^2-1}$
(c) $\frac{4x^2-7x-12}{x(x+2)(x-3)}$
(d) $\frac{x^2+2x-1}{x^3-x}$